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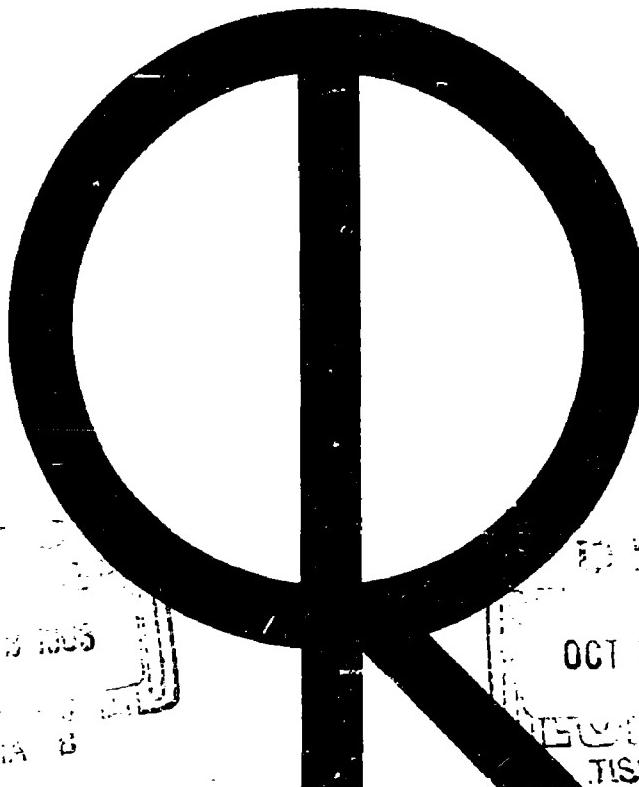
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## Project Came Bridge Technical Memorandum

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) DETECTION STATION  
OPTIMIZATION - I ,

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CAME BRIDGE MEMORANDUM No. 5

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DETECTION PHYSICS LABORATORY  
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES  
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Project CAME BRIDGE is the unclassified nickname which identifies the USAF research program investigating long range (over-the-horizon) sensing of information of rocket launchings. This program includes theoretical studies, laboratory experiments and field observations of pertinent physical phenomena and sensing techniques. Project CAME BRIDGE is conducted by the Detection Physics Laboratory, Air Force Cambridge Research Laboratories, L.G. Hanscom Field, Bedford, Massachusetts and a team of supporting contractors, as part of the work documented in Projects 4649 "Over-the-Horizon Detection" and 4691 "Recognition".

## I. INTRODUCTION

One problem that arises in implementing a network for the detection of specific operations over an area of interest is the optimum placement of a given number of detection stations. Another question that is related to this station placement problem is the selection of the economically optimum number of detection stations. The solution of these two problems has many applications; for example, in the selection and placement of (1) missile launch detection stations, (2) nuclear test detection stations, (3) disarmament control stations, and (4) aircraft detection stations.

This report will set up the formalism for an attack on these two problems, discuss a suggested approach to the problem, and then present some preliminary results that have been produced.

Basically, the problem to be attacked is the following: We are given an arbitrary area within which the enemy is likely to produce an event that we would like to detect. Our detection mechanism shall be detection stations that are set up within this area; the effectiveness of each station in detecting an event shall, of course, decrease with its distance from the event. If now, we are given  $n$  stations to place in the area of Figure 1, the first question posed above is the optimum placement of these  $n$  stations within the area so that their effectiveness in detecting enemy events is a maximum.

The second question is one of cost. As we place more and more stations within the area, our effectiveness in detecting enemy events will increase in general; but it can not, of course, increase indefinitely since we can do no better than detect all enemy events with unity probability. On the other hand there will be a definite cost associated with placing the stations within the area, and as we place more and more stations within the area we shall eventually reach a point of diminishing returns. The question, then is one of determining the optimum trade-off between the effectiveness of the network of stations and its cost.

## II. PROBLEM FORMAT

The following notation will be used in the preliminary analysis of the problem:

$n$  = the number of detection stations.

$p(r)$  = the probability that a station will detect an enemy event if it occurs a distance  $r$  from the station (assumed independent of location and orientation for this preliminary analysis).

$q(r) = 1 - p(r)$  It is assumed throughout that:

$$\frac{dq(r)}{dr} > 0 \text{ for all } r \quad (1)$$

i.e., the probability of no detection increases with distance.

$w(r) = \log q(r)$

$x_i$  = the  $x$  coordinate of the  $i$ th station

$y_i$  = the  $y$  coordinate of the  $i$ th station

$X = \{x_1, x_2, x_3, \dots, x_n\}$  the set of  $x$  coordinates for the  $n$  stations

$Y = \{y_1, y_2, y_3, \dots, y_n\}$  the set of  $y$  coordinates for the  $n$  stations

$C_s(n)$  = the cost of implementing a network of  $n$  stations (assumed independent of  $X$  and  $Y$  for this initial investigation.)

$C_d(V) =$  the cost to us if we can detect an enemy event with probability  $(1 - e^{-V})$

As implied by the above notation, the effectiveness criterion that we shall use is the probability of detecting an enemy event. This is a desirable criterion in that it conforms with the intuitive ideas of a "good" detection system.

We shall assume in this report that the enemy will know where our detection stations are located as well as how effective they are, i.e., he knows  $X$ ,  $Y$ , and  $p(r)$ . If he has this information, then he will certainly make sure that his

event occurs at a point where its probability of not being detected by us is a maximum.

Now if there are  $n$  stations each operating independently of the other, the probability  $Q(x, y)$  that an event occurring at  $(x, y)$  will not be detected is just the product of the probabilities that each of the stations does not detect it:

$$Q(x, y) = \prod_{i=1}^n q(\sqrt{(x-x_i)^2 + (y-y_i)^2}) \quad (2)$$

Taking the logarithm of both sides of Eq. 2 we have

$$W(x, y) = \sum_{i=1}^n \log q(\sqrt{(x-x_i)^2 + (y-y_i)^2}) = \sum_{i=1}^n w(\sqrt{(x-x_i)^2 + (y-y_i)^2}) \quad (3)$$

And by the argument of the preceding paragraph we know that the enemy will cause its event to occur at that  $(x, y)$  where  $W(x, y)$  is a maximum. Our problem is to place the  $n$  stations such that this maximum value of  $W(x, y)$  is a minimum. In other words, if we define:

$$\max_{X, Y} M(X, Y) = \min_{x, y} W(x, y) \quad (4)$$

where  $(x, y)$  ranges over all points of the area in question; then our problem is to find that  $X$  and  $Y$ , such that  $M(X, Y)$  is a minimum. This minimum value of  $M(X, Y)$  will be labeled  $V(n)$ :

$$V(n) = \min_{X, Y} M(X, Y) = \min_{X, Y} \max_{x, y} W(x, y) \quad (5)$$

Thus we are faced with a typical game-theory minimax problem. This is not surprising since the problem as posed here fits quite naturally into the game theory formalism. The fact that it is a perfect information game insures that the best strategies will be pure strategies (i.e. not random).

Once we have found  $V(n)$  for all  $n$  as well as the optimum placement of the detection stations, the question as to how many detection stations to use can be answered. Obviously, one can place detection stations infinitely dense over the area and insure that the probability of missing a detection will be zero. This would require infinite resources, however, so one assumes the existence of a

cost function  $C_d(V)$ , that is, the cost to us if our probability of missing a detection is  $e^{-V}$  (assuming, of course, that the enemy event will occur at the best location for the enemy).

The total cost to us for an  $n$  station network is then:

$$C_t(n) = C_s(n) + C_d(V(n)) \quad (6)$$

For small  $n$  the cost of the detection stations will be small,  $V(n)$  will be large, and the  $C_d$  term in Eq. 5 will predominate, while for large  $n$ ,  $V(n)$  will be small and the cost of the stations,  $C_s$ , will predominate. The optimum value of  $n$  occurs when  $C_t$  is a minimum,

$$C_t(n_{opt}) \leq C_t(n_{opt} \pm 1) \quad (7)$$

The cost analysis in Eqs. 6 and 7 assumes that  $C_s(n)$  is independent of the location of the detection stations and their relative positions. If this is not true, then the problem is much more complicated, since the optimum placement of the stations must now take into account this additional cost consideration.

### III. APPROACH

In the solution of this problem we shall proceed from the simplest possible situations to the more realistic cases, being careful at each point to list the assumptions and approximations that have been made.

The first case to be analyzed will be the placement of  $n$  identical stations on a unit line; that is, it is assumed that the event will occur some place on the unit line and we would like to find the placement of  $n$  stations such that:

$$\max_x W(x) = \max_x \sum_{i=1} w(x-x_i) \quad (8)$$

is a minimum. In Section 4 we shall list several necessary conditions for this optimum placement. Sufficiency, existence, and uniqueness remain to be proved. These conditions also apply for the non-identical station situation.

Some preliminary remarks are also made in Section 4 concerning the two-dimensional situation for specially symmetrical areas.

Once these simpler models of the problem have been solved, there are several directions that can be taken in making the above problem more realistic. Some of them are:

1. The detection station effectiveness--or probability of detection--may be a function not only of distance from the event, but also the angle from some preferred direction. In this case, one must consider not only the placement of the stations, but also their orientation.

2. In the initial analysis symmetrical areas will be used, but ultimately, one must extend the problem solutions to arbitrary areas.

3. There are instances in which there will be constraints on the location of detection stations. As examples, it may be necessary for stations to lie in a straight line, or to be within a certain minimum distance of one another, or perhaps not to lie in certain areas.

4. While we will be concerned with identical detection stations initially, there is the possibility that the detection stations may not be identical.

5. The effectiveness of a station may also vary with its location within the area.

6. The simplest assumption for the cost of the network of stations is a linear one, i.e.  $C_s(n) = nc$ . There may be, however, more complicated station cost functions. Furthermore, the cost of a station may also be a function of its placement, and even the relative location of other stations.

## IV. SOME PRELIMINARY RESULTS

### 4.1 The One Dimensional Case

In this section we shall consider the problem of the placement of  $n$  detection stations on the unit line.

#### 4.1.1 Two Stations

Let us consider first of all the placement of two stations on the unit line when:

$$\frac{d^2 w(r)}{dr^2} > 0 \text{ for all } r \quad (9)$$

This situation is illustrated in Figure 2. Symmetry of the situation and the identical nature of the stations require that  $x_2 = 1 - x_1$  for any fixed value of  $x_1$ . If it were not true, then one of the boundary values of  $W(x)$  ( $W(0)$  and  $W(1)$ ) would be greater than the other and so  $M(x_1, x_2)$  would not be a minimum.

As shown in Figure 2,  $W(x)$  will have its maximum value at  $x=0$  and  $x=1$ . Furthermore, Condition 9 requires that:

$$\left| \frac{\partial W(0)}{\partial x_2} \right| = \left| \frac{\partial w(x-x_2)}{\partial x_2} \right|_{x=0} > \left| \frac{\partial w(x-x_1)}{\partial x_1} \right|_{x=0} = \left| \frac{\partial W(0)}{\partial x_1} \right| \quad (10)$$

$$\left| \frac{\partial w(x-x_1)}{\partial x_1} \right|_{x=1} > \left| \frac{\partial w(x-x_2)}{\partial x_2} \right|_{x=1}$$

for  $x_2 > x_1$

Therefore, if  $x_1$  and  $x_2$  are moved closer to the center of the line (with  $x_1 = 1 - x_2$ ) the values of  $W(1)$  and  $W(0)$  will decrease. Thus  $M(x_1, x_2)$  will have its minimum value at  $x_1 = x_2 = 1/2$ , and  $V(2) = 2w(1/2)$  for this case.

The next case to be considered is the placement of two stations when:

$$\frac{d^2 w(r)}{dr^2} < 0 \text{ for all } r \quad (11)$$

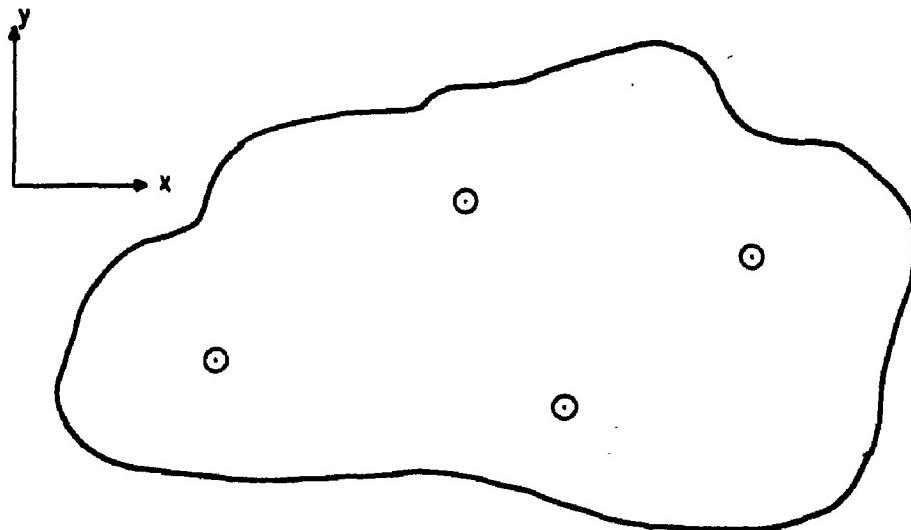


FIGURE 1. STATION PLACEMENT EXAMPLE FOR  $n=4$

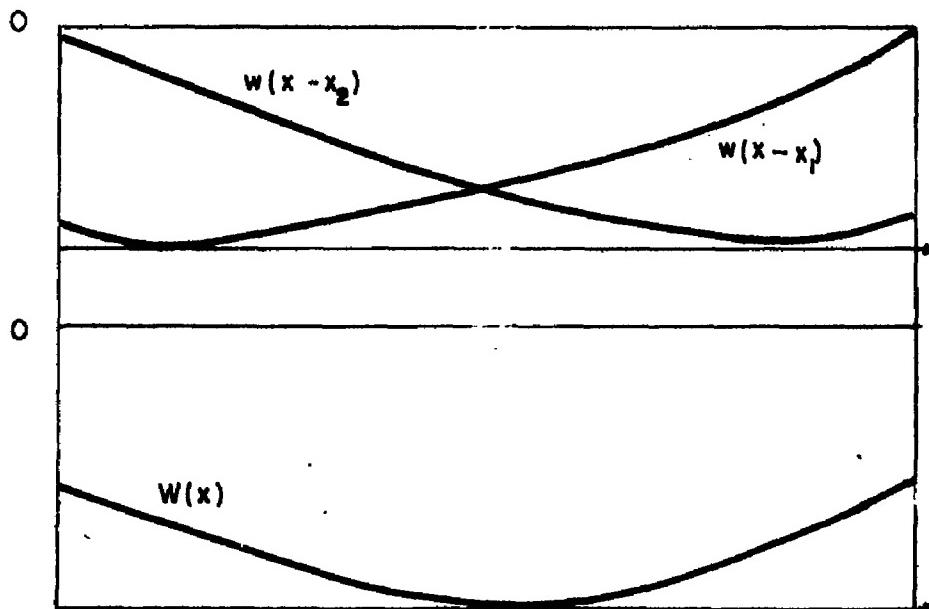
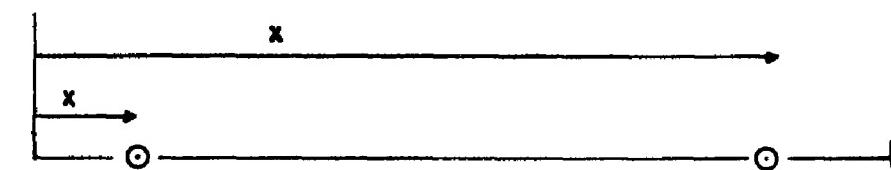


FIGURE 2. ONE DIMENSIONAL CASE,  $n=2$ ,  $\frac{d^2w(r)}{dr^2} > 0$

(Although the results for this situation can be applicable to the mixed case in which Condition 9 and Condition 11 apply for different ranges of  $r$ , we shall concern ourselves principally with these two cases in this report. The case where Condition 11 applies with the equal sign is the linear case in which  $M(x_1, x_2)$  is independent of the location of the stations, as long as  $x_1 = 1 - x_2$ ).

The situation for Condition 11 is shown in Figure 3; the same argument that we used before applies here for making  $x_1 = 1 - x_2$ . As illustrated in Figure 3,  $W(x)$  will always have a local maximum at  $x=1/2$  as long as  $x_1 = 1 - x_2 \neq 1/2$ . If  $x_1 = 0$  and  $x_2 = 1$ , this maximum at  $x=1/2$  will also be the absolute maximum for all  $x$  between 0 and 1; i.e.,  $M(0, 1) = 2w(1/2)$ . Now if  $x_1$  and  $x_2$  are moved toward the center of the unit line at the same speed, the value of  $W(1/2)$  will decrease. On the other hand Condition 11 assures us that the movement of  $x_1$  will have more effect on  $W(0)$  than the movement of  $x_2$  as long as  $x_2 > x_1$ ; more precisely:

$$\left| \frac{\partial W(0)}{\partial x_1} \right| = \left| \frac{dw(x_1)}{dx_1} \right| > \left| \frac{dw(x_2)}{dx_2} \right| = \left| \frac{\partial W(0)}{\partial x_2} \right| \quad (12)$$

for  $x_2 > x_1$ . Therefore, as we move  $x_1$  and  $x_2$  toward the center, the value of  $W(0) = W(1)$  will increase as  $W(1/2)$  decreases. At some value for  $x_1$ , the values of  $W(0)$ ,  $W(1)$ , and  $W(1/2)$  will all be equal; and if  $x_1$  and  $x_2$  ( $=1-x_1$ ) are moved any further toward the center, the maximum value of  $W(x)$  will occur at  $x=0$  and 1 instead of at  $x=1/2$ . And since  $W(0)$  will be increasing,  $M(x_1, 1-x_1)$  will cease decreasing and start to increase. Thus,  $M(x_1, 1-x_1)$  will have its minimum value at the point where  $W(0) = W(1) = W(1/2)$ .

In equation form, the optimum value of  $x_1$  is the solution of the equation:

$$W(0) = W(1/2) \\ w(x_1) + w(1-x_1) = 2w\left(\frac{1}{2} - x_1\right) \quad (13)$$

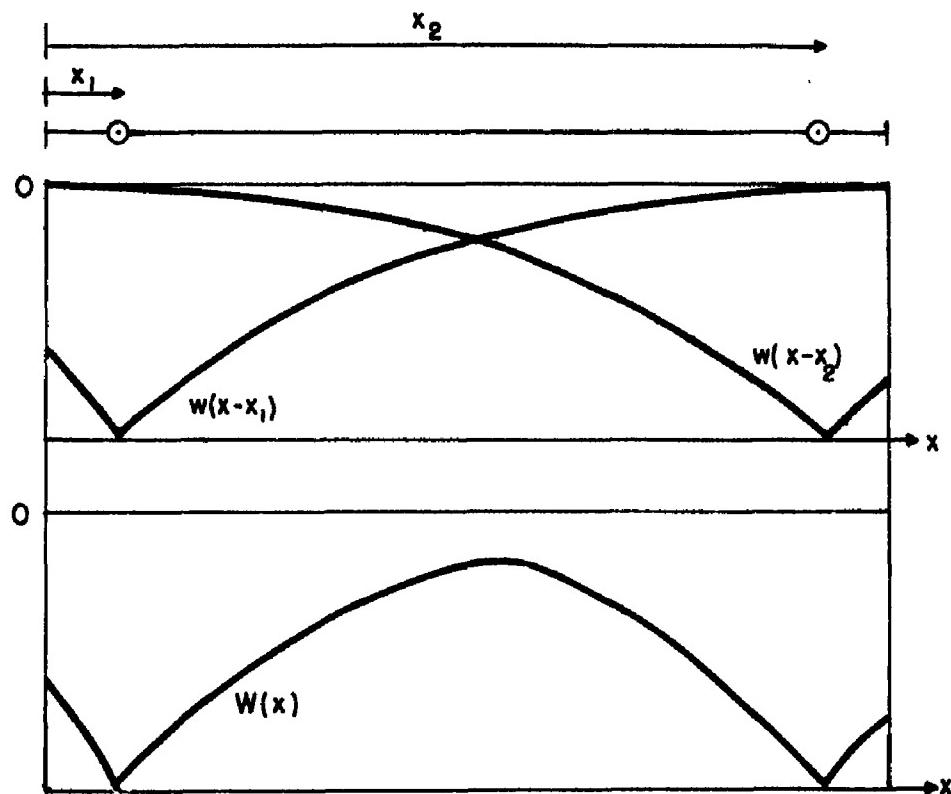


FIGURE 3. ONE DIMENSIONAL CASE,  $n=2$ ,  $\frac{dw(r)}{dr^2} > 0$

Two graphical methods for solving Eq. 13 are shown in Figures 4 and 5. In Figure 4, the intersection of  $w(x) + w(1-x)$  with  $2w(\frac{1}{2} - x)$  is found, while Figure 5 portrays a trial and error method in which values of  $x$  are tested to see whether the average of  $w(x)$  and  $w(1-x)$  is equal to  $w(\frac{1}{2} - x)$ . In Figure 5,  $x_1^{(1)}$  is an initial trial that was incorrect;  $x_1^{(2)}$  is a successful trial.

For the non-identical station situation in which Condition 11 still holds for both stations, the optimum placements are somewhat more difficult to obtain. In this case, we are no longer allowed to assume that  $x_2 = 1 - x_1$ . In general, however, if  $W(x)$  has a local maximum somewhere between the two stations, then a necessary condition for optimum placement is for:

$$W(0) = W(1) = W(x_0) \quad (14)$$

where  $x_0$  is the location of the local maximum. The arguments for obtaining these conditions are exactly the same as those used in deriving the solution for the two-identical-station case. Thus, if a local maximum exists for some  $x_1 < x_0 < x_2$  then a possible optimum placement is the solution of the equations:

$$w_1(x_0 - x_1) + w_2(x_0 - x_2) = w_1(x_1) + w_2(x_2) = w_1(1 - x_1) + w_2(1 - x_2) \quad (15)$$

where  $w_1(r)$  and  $w_2(r)$  are the logarithms of the probability of a missed detection for the two stations, and  $x_0$  is the solution of the equation:

$$\sum_{i=1}^n w_i^*(x_0 - x_i) = 0 \quad (16)$$

The reason behind the cautious wording in the above paragraph is that there may be situations in which Condition 11 is satisfied for both stations, but for which  $W(x)$  has no local maximum for certain values of  $x_1$  and  $x_2$ . As an example consider the situation in Figure 6a in which  $w_1^*(r) > w_2^*(0+)$  for  $r < r_0$ . If one then plots  $W^*(x)$  as the sum of  $w_1^*(x - x_1) + w_2^*(x - x_2)$  as in Figure 6b, it is obvious that there will be no local maximum between  $x_1$  and  $x_2$  if  $(x_2 - x_1) < r_0$ . Thus, if one starts with  $x_1 = 0$  and  $x_2 = 1$  and moves them toward

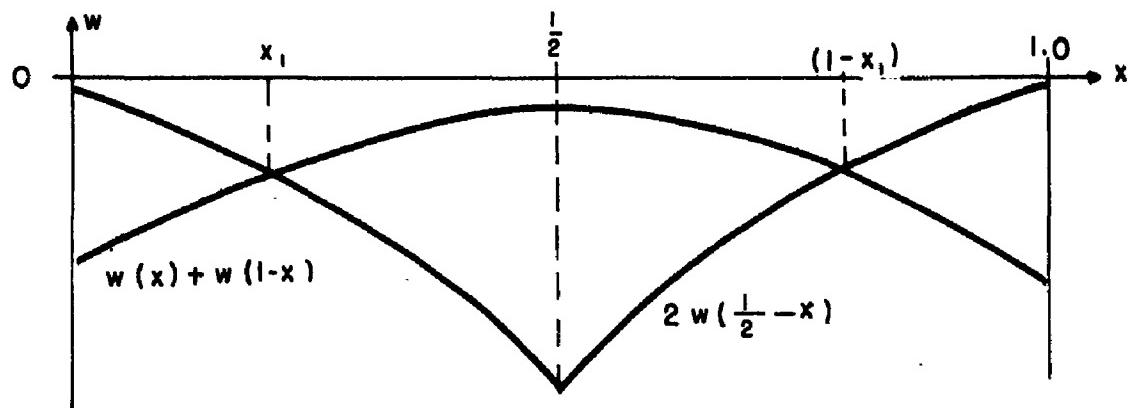


FIGURE 4. SOLUTION OF STATION PLACEMENT FOR 2 IDENTICAL STATIONS-I

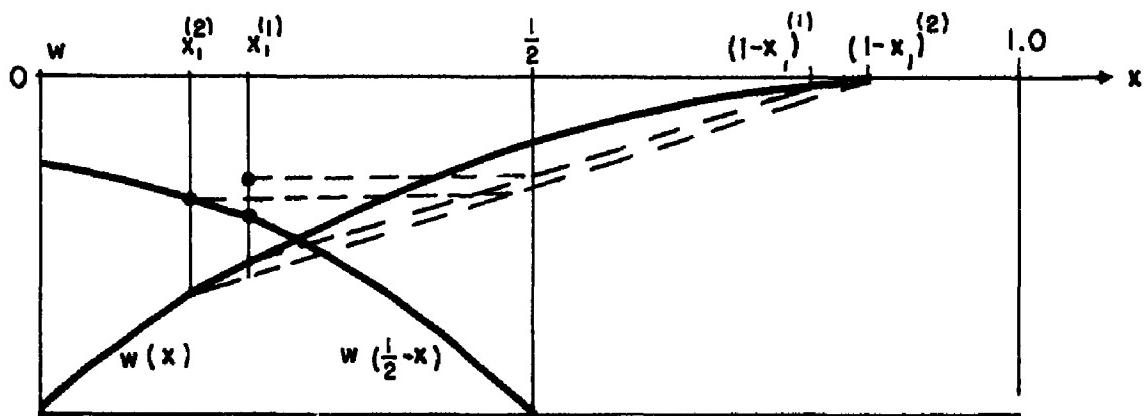


FIGURE 5. SOLUTION OF STATION PLACEMENT FOR 2 IDENTICAL STATIONS-II

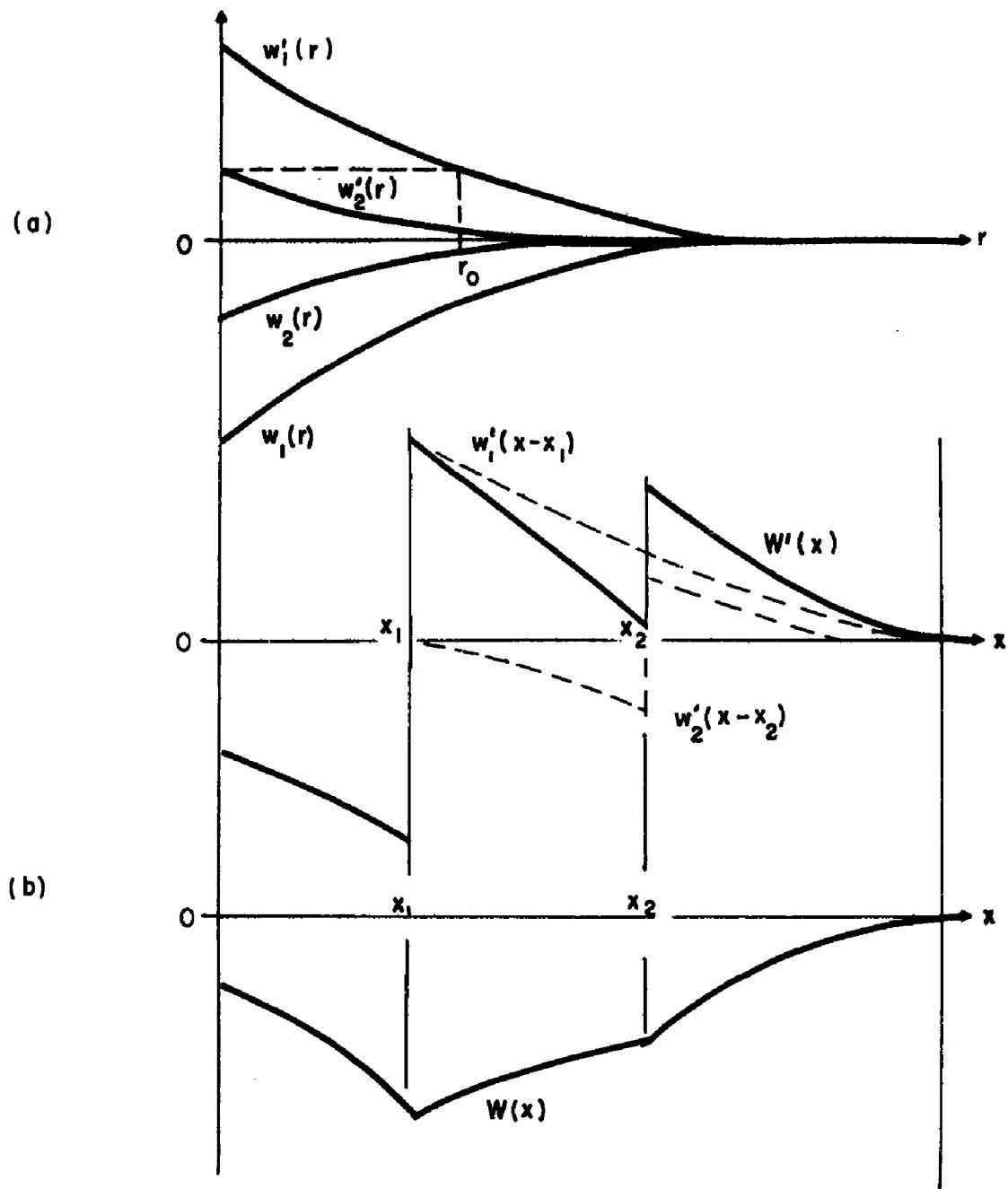


FIGURE 6. THE TWO NON-IDENTICAL STATION SITUATION FOR NO LOCAL MAXIMUM

the center under the constraint that  $W(0) = W(1)$ , it may happen that the local maximum disappears before Eqs. 15 and 16 are satisfied. If this occurs then one must look for another solution to the problem.

This "other solution" can be illustrated by considering the case in Figure 6a when  $r_0 > 1$ . For this case there will never be a local maximum between  $x_1$  and  $x_2$  and so we must only consider the end points. A necessary condition for the solution is that the values of  $W$  at these two end points be equal. If they were not, then  $x_1$  and  $x_2$  could be moved closer to the end point with the maximum value of  $W$ , thus reducing  $M$  even more. Therefore, our problem is to find the location of  $x_1$  and  $x_2$  such that  $W(0) = W(1)$  is a minimum.

To find the solution let us start with  $x_1 = x_2 = 1/2$  and move the stations away from the center under the constraint that  $W(0) = W(1)$ . Due to the fact that Condition 11 applies to both stations  $W(0)$  and hence  $M(x_1, x_2)$  will decrease until a station is at one of the ends of the line; for example, if  $w_2^*(0+) < w_1^*(1)$  and if we arbitrarily choose  $x_2 > x_1$  then the minimum value of  $M(x_1, x_2)$  will occur at:

$$x_2 = 1$$

$x_1$  = the solution of the equation:  $w_1(x) - w_1(1-x) = w_2(0) - w_2(1)$  (17)

Therefore, the complete solution for the optimum placement of two stations (whether identical or not) is the following:

1. Start both stations at  $x_1 = x_2 = 1/2$  and move them outward one station in each direction -- keeping  $W(0) = W(1)$ .
2. If a maximum value arises between the two stations that equals  $W(0)$ , then stop; this is the optimum placement of the stations.
3. If no maximum of this height materializes, stop as soon as one of the stations is at the end of the unit line, and this is the solution.

#### 4.1.2 n Identical Stations

The arguments of the previous section can be applied to the  $n$  identical station situation to yield some necessary conditions for the optimum placement of the  $n$  stations on the unit line. First of all let us dispose of the case in which Condition 9 applies.

As in the two identical station case, the stations must be placed symmetrically

about the  $x = 1/2$  point; otherwise, the values of  $W(x)$  for one half of the unit line will predominate over the other half, and those maximum values can be decreased by making the placement of the stations more symmetrical. Therefore, if we number the station locations  $\{x_1, x_2, \dots, x_n\}$  with  $x_{i+1} > x_i$ , the identical nature of the stations forces:

$$x_i = 1 - x_{n-i+1} \text{ for } 1 \leq i \leq n \quad (18)$$

Now consider the location of some station and its mate for the Condition 9 situation (see Figure 7). If  $x_i < 1/2$ , then as  $x_i$  and  $x_{n-i+1}$  are moved closer to the center of the unit line (under the constraint of Eq. 18) Condition 9 will force the value of  $W(x)$  to decrease for all values of  $x$ . This is true because for  $x < 1/2$ ,  $w(x-x_{n-i+1})$  will predominate while  $w(x-x_i)$  will predominate for  $x > 1/2$ . Therefore, if Condition 9 applies, the optimum placement of  $n$  stations is for:

$$x_i = 1/2 \quad 1 \leq i \leq n \quad (19)$$

Before proceeding to the Condition 11 situation, let us investigate the implications of these limitations on  $d^2 w/dr^2$ . Figure 8 shows examples of  $w(r)$ ,  $q(r)$ , and  $p(r)$  for the three cases in which  $d^2 w/dr^2$  is positive, zero, and negative for all  $r$ . As stated earlier, we shall not consider situations in which  $d^2 w/dr^2$  has different polarities for various ranges of  $r$ . For the case in which  $d^2 w/dr^2$  is zero,  $p(r)$  has the form

$$p(r) = \begin{cases} 1 - Ae^{Br} & r < -\frac{1}{B} \ln A \\ 0 & r > -\frac{1}{B} \ln A \end{cases} \quad (20)$$

The restrictions that  $p(r)$  be a probability (i.e., between 0 and 1) and that  $dq/dr$  be positive require that:

$$A \leq 1$$

$$B \geq 0$$

For the case in which  $d^2 w(r)/dr^2$  is negative, the following three conditions are necessary for the optimum placement of  $n$  identical stations:

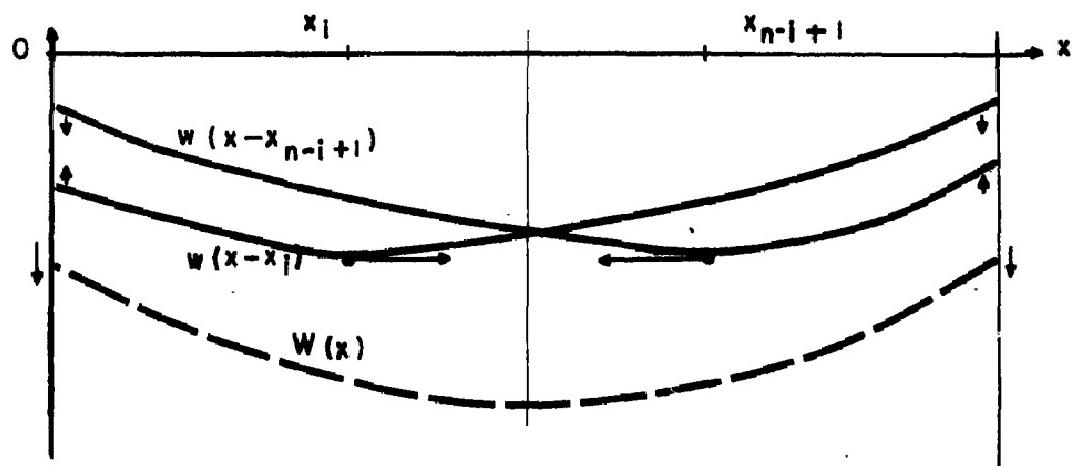


FIGURE 7. ONE DIMENSIONAL CASE FOR  $n$  IDENTICAL STATIONS

$$\text{AND } \frac{d^2w}{dr^2} > 0$$

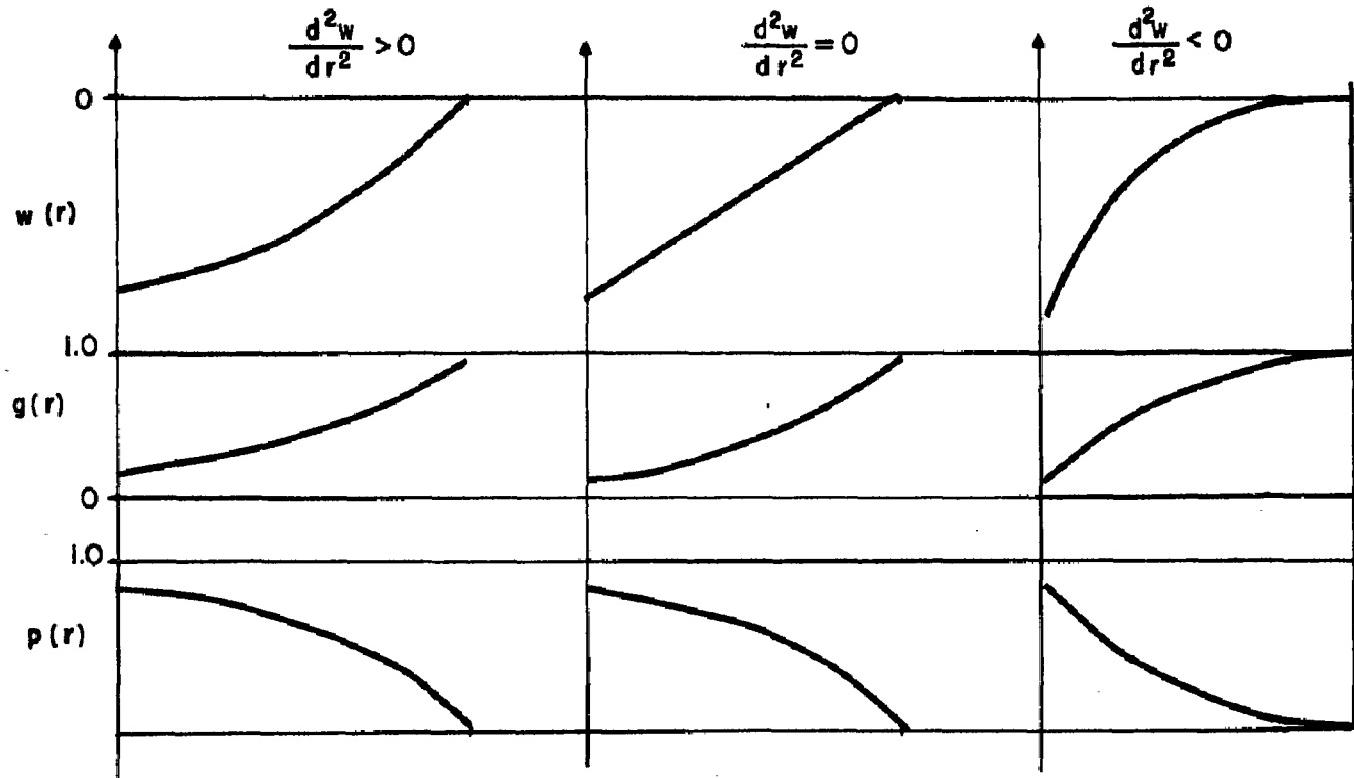


FIGURE 8.  $w(r)$ ,  $g(r)$  AND  $p(r)$  FOR THE THREE CASES :

$$\frac{d^2w(r)}{dr^2} > 0, = 0, \text{ AND } < 0.$$

1.  $x_i = 1 - x_{n-i+1}$  The symmetry argument for this condition is the same as the above argument for  $d^2w(r)/dr^2$  positive. (We shall assume that  $x_{i+1} \geq x_i$  for convenience from here on.)

2. The value of  $W(x)$  at the end points must be equal to the maximum value of  $W(x)$  over the entire range of  $x$ ; i.e.,

$$W(0) = W(1) = M(x_1, x_2, \dots, x_n) = M(X) \quad (21)$$

The argument for this (and for the next) is similar to the one used for only two identical stations.

Consider the situation in which  $W(0) = W(1)$  is not equal to the maximum of  $W(x)$ ; i.e. the situation in which  $W(0) = W(1) < M(X)$ . (Figure 9) For this situation it will in general be possible to move  $x_1$  and  $x_n$  (the two outermost stations) closer to the midpoint of the unit line as shown in Figure 9. Because of Conditions 1 and 11 this movement of the two stations will cause  $W(0) = W(1)$  to increase while the maximum of  $W(x)$  in the interval between  $x_1$  and  $x_n$  will decrease. Thus, this movement of the two stations will decrease  $M(X)$  (since this maximum occurs between  $x_1$  and  $x_n$ ; see Figure 9), and so the original placement of the stations was not optimum, since there are other placements that have a lower value of  $M(X)$ .

3. For all  $1 \leq i < n$  the maximum value of  $W(x)$  in the interval  $(x_i, x_{i+1})$  must be equal to  $M(X)$ .

Consider the situation in Figure 10 in which  $W(x)$  has its maximum value at  $x = x_{i+1}$  (an equally likely situation is for  $W(x)$  to have a local maximum for some  $x_i < x < x_{i+1}$ ). Unless  $x_{i+1} = 1$ , it is obvious that  $W(x_{i+1})$  is not equal to  $M(X)$  since there are values of  $x$  just to the right of  $x_{i+1}$  for which  $W(x) > W(x_{i+1})$ . Since  $W(x) < M(X)$  for all  $x_i < x < x_{i+1}$ ,  $M(X)$  will not be increased by increasing  $W(x)$  in this interval. Therefore, let us move  $x_1$  and  $x_{n-i+1}$  toward the endpoints and  $x_{i+1}$  and  $x_{n-i}$  toward the center all at the same speed.

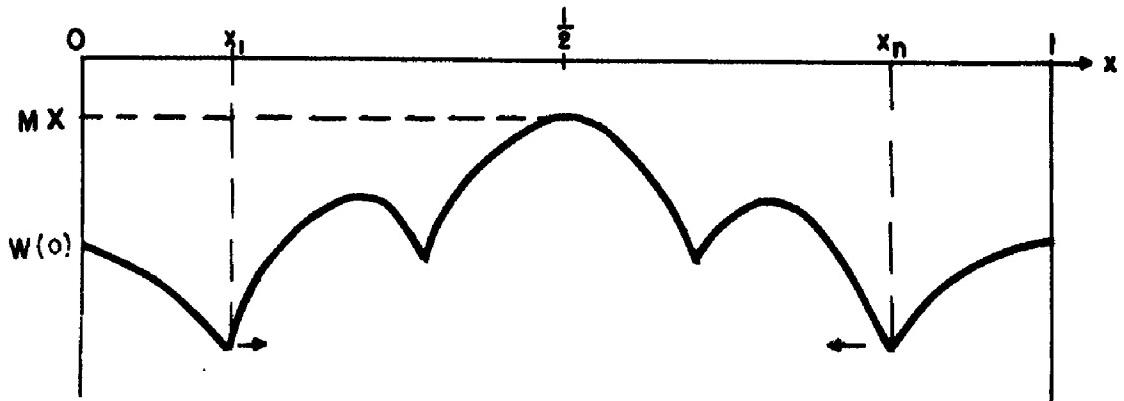
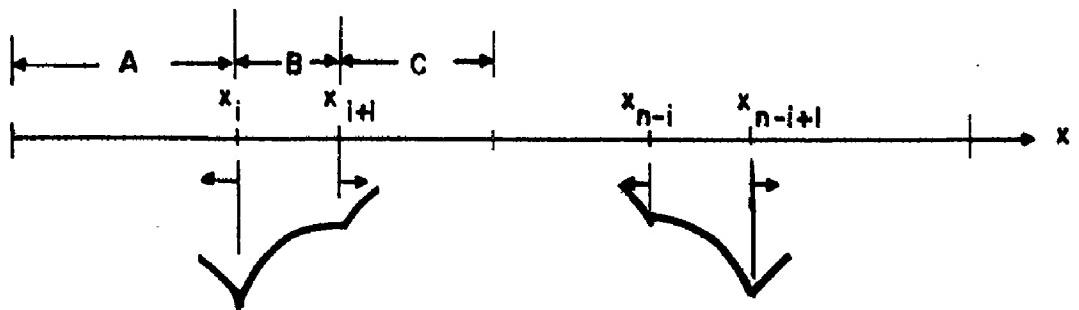


FIGURE 9.



REGION	EFFECT OF STATION MOTION					NET EFFECT
	$x_i$	$x_{i+1}$	$x_{n-i}$	$x_{n-i+1}$		
A	D	I	D	I		DECREASES
B	I	I	D	I		INCREASES
C	I	D	D	I		DECREASES

D=DECREASES

I=INCREASES

FIGURE 10.

As shown in Figure 10 these four motions will cause  $W(x)$  to increase in the  $\{x_i, x_{i+1}\}$  and  $\{x_{n-i}, x_{n-i+1}\}$  intervals and decrease elsewhere. And since the maximum value of  $W(x)$  occurs outside the above two intervals, the net result is a decrease in  $M(X)$ . Thus, the optimum placement of the  $n$  stations cannot occur unless the third condition above is satisfied.

It is worth mentioning here that the value of  $W(x)$  at a station site can equal  $M(X)$ , [i.e. the case in which  $W(x_i) = M(X)$ ] only when  $x_1 = 0$ . In this case the first and second necessary conditions force  $W(x_1) = W(0) = W(x_n) = W(1) = M(X)$ , and so the third condition is automatically satisfied for  $i = 1$  and  $i = n-1$ . It can be easily proved that this is the only case in which  $W(x_i) = M(X)$ :

As shown in Figure 6, the slope of  $w(x-x_i)$  is discontinuous at  $x = x_i$ ; it jumps from some negative value to the positive value at that point. This is a necessary by product of Conditions 1 and 11 and the fact that  $p(x) = p(-x)$ .

Now for  $W(x)$  to have a local maximum at  $x = x_i$ , the slope of  $W(x)$  must pass through zero (continuously or discontinuously) from positive to negative at  $x = x_i$ . But since  $dw(x-x_i)/dx$  has a jump in the opposite direction at  $x = x_i$ , the conditions for  $W(x)$  to have a local maximum at  $x = x_i$  are impossible as long as  $w(x)$  has a continuous derivative for all  $x \neq 0$ . This argument obviously does not apply if  $x_i = 0$  or 1. Hence for any placement of the stations,  $W(x_i)$  must be less than  $M(X)$  for all  $i$  that have  $x_i \neq 0$  or 1.

While there is no logical reason why the optimum placement of  $n$  stations will not have  $x_1 = 0$  and  $x_n = 1$ , intuitively one would not expect this to occur for small  $n$ , since only half of the stations ranges are being used if they are placed at the ends of the line.

While we have given three necessary conditions for the optimum placement of  $n$  stations, we have not proved the sufficiency of these three conditions, nor have we proved that Conditions 1 and 11 insure the existence of a unique solution that satisfies these conditions. These three things must still be proved.

It is worth noting that the solution in Section 4.1.1 for two non-identical stations still satisfies the second and third above conditions. In fact the second and third conditions are also necessary for the optimum placement of  $n$  non-

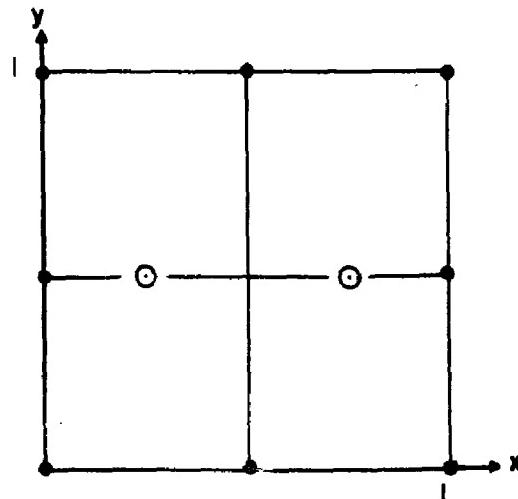
identical stations. The proofs proceed exactly as they did for the  $n$  identical station problem, with the exception that the restriction that  $x_i = 1 - x_{n-i+1}$  is relaxed. This allows one the freedom to move only  $x_1$  (or  $x_n$ ) in the proof of the second condition and only  $x_i$  and  $x_{i+1}$  in the proof of the third condition.

#### 4.2 Future Work

With the results of Section 4.1 as a beginning, the following are the problems that will be attacked next:

- A. The programming of the " $n$  stations on the unit line" problem on the DX-1 computer to determine  $V(n)$  and  $X_{opt}$ .
- B. The extension of the results of Section 4.1 to the situation in which  $p(r)$  is a function of location.
- C. The extension of the results of Section 4.1 to the situation in which the cost of the station sites is a function of their location.
- D. The extension of the results of Section 4.1 to the two dimensional situation. The solutions of some simple cases of symmetrical areas are shown in Figure 11.

2 IDENTICAL STATIONS



CONDITIONS FOR OPTIMUM PLACEMENT

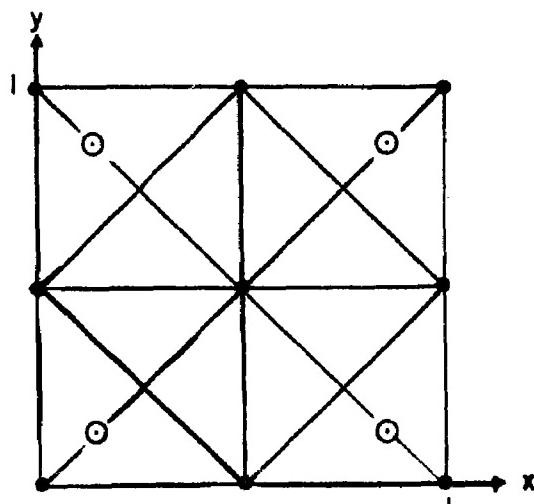
$$y_1 = y_2 = \frac{1}{2}$$

$$W(0,0) = W\left(\frac{1}{2},0\right) = W(1,0) =$$

$$W(0,1) = W\left(\frac{1}{2},1\right) = W(1,1) =$$

$$M(x_1, x_2, \frac{1}{2}, \frac{1}{2})$$

4 IDENTICAL STATIONS



$$x_1 = y_1, x_3 = y_3$$

$$x_2 = 1 - y_2, x_4 = 1 - y_4$$

$$\text{EITHER: } W(0,0) = W(0,1) =$$

$$W(1,0) = W(1,1) =$$

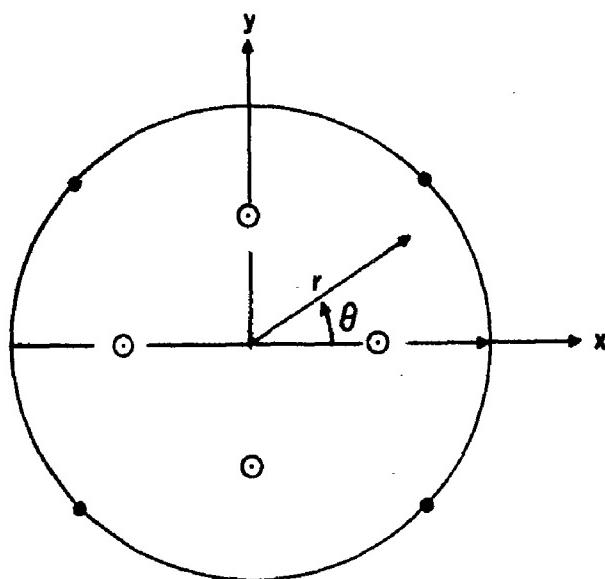
$$W\left(\frac{1}{2}, \frac{1}{2}\right) = M(X, Y)$$

$$\text{OR: } W(0,0) = W(0,1) = W(1,0) = W(1,1) =$$

$$W\left(0, \frac{1}{2}\right) = W\left(\frac{1}{2}, 1\right) = W\left(1, \frac{1}{2}\right) = W\left(\frac{1}{2}, 0\right) =$$

$$M(X, Y)$$

4 IDENTICAL STATIONS



$$r_0 = r_1 = r_2 = r_3$$

$$\theta_1 = \frac{\pi}{2}$$

$$W(0,0) = W\left(1, \frac{\pi}{4}\right) = W\left(1, \frac{3\pi}{4}\right) =$$

$$W\left(1, \frac{5\pi}{4}\right) = W\left(1, \frac{7\pi}{4}\right) = M(R, \Theta)$$

FIGURE 11

**UNCLASSIFIED**

**UNCLASSIFIED**